Assignment 5

1. Consider the metric space $\mathcal{C}[0,1]$ with sup norm. Let $A = \{f \in \mathcal{C}[0,1] \mid \|f\| \leqslant 1\}$. Show that:
   (a) A is closed and bounded subset of $\mathcal{C}[0,1]$.
   (b) A is not compact.

2. For a uniformly continuous function f on $\mathbb{R}$, define $\Lambda_f = \inf\{A \mid |f(x)| \leqslant A + B|x|\}$. Show that $\Lambda_f = |f(0)|$. Also find a uniformly continuous function g on $\mathbb{R}$ such that $|g(x)| \leqslant |g(0)| + B|x|$ does not hold true for any $B > 0$.

3. Suppose E be a dense subset of a metric space X and $f : E \to \mathbb{R}$ be a uniformly continuous function. Show that f has a unique continuous extension from E to X, that is, there exists a unique continuous function $\tilde{f} : X \to \mathbb{R}$ such that $\tilde{f}(x) = f(x)$ for all $x \in E$.

4. For a compact set E of R, and a real-valued function f defined on E, define $\mathrm{Graph}(f) = \{(x, f(x)) \mid x \in E\}$. Assuming that $\mathbb{R}^2$ is a metric space with $d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, show that f is continuous on E if and only if $\mathrm{Graph}(f)$ is compact in $\mathbb{R}^2$.

5. Call a function $f : \mathbb{R} \to \mathbb{R}$ open if $f(V)$ is open whenever V is open in $\mathbb{R}$. Prove that a continuous open function from $\mathbb{R}$ to $\mathbb{R}$ is monotonic.

Bonus Questions (Need not to be submitted)

6. Show that there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that f is open but not monotonic.

7. Suppose $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that for each $\alpha \in \mathbb{R}$, $\lim_{n\to\infty} f(\alpha n) = 0$. Show that $\lim_{|x|\to\infty} f(x) = 0$.